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Particular Solutions of Forced Generalized Airy Equations

ROBERT M. MIURA*

Department of Mathematics and Institute of Applied Mathematics, University of British Columbia, Vancouver, British Columbia V6T 1Y4, Canada

AND

CHARLES G. LANGE†

Department of Mathematics, University of California, Los Angeles, California 90024

1. INTRODUCTION

Differential equations with delayed forcing terms arise naturally in the explicit solution of differential-delay equations [1]. In our present studies of singular perturbation analyses of boundary-value problems for second-order differential-difference equations with a turning point (see [2] for earlier results), we have had to deal with forcing terms consisting of shifted Airy and Lommel functions and their derivatives. One example of the type of equation which arises is the inhomogeneous Airy equation (see the Appendix for a derivation)

$$\begin{aligned} y'' - xy = {} & c_1 Ai(x-a) + c_2 Bi(x-a) + c_3 L(x-a) \\ & + d_1 Ai'(x-a) + d_2 Bi'(x-a) + d_3 L'(x-a), \end{aligned} \tag{1}$$

where a is a constant $(0 < a < \infty)$; c_i, d_i, $i = 1, 2, 3$, are constants; Ai, Bi are Airy functions; and L is a Lommel function (see [3]) satisfying

$$L'' - xL = 1. \tag{2}$$

In this paper, we consider generalizations of (1) in the form

$$\mathscr{L}y - xy = cf(x), \tag{3}$$

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where $\mathscr{L}$ is the kth-order linear differential operator with constant coefficients given by

$$\mathscr{L} \equiv \sum_{j=0}^{k} \alpha_j \frac{d^j}{dx^j}, \qquad \alpha_j = \text{constant}, \quad j = 0, 1, ..., k. \tag{4}$$

We find exact particular solutions for special classes of functions $f(x)$.

In Section 2, we obtain exact solutions for the general operator $\mathscr{L}$. In Section 3, we restrict $\mathscr{L} = \mathscr{D}^k$, where $\mathscr{D} \equiv d/dx$, and obtain explicit solutions for a wider class of functions $f(x)$.

2. Forced Generalized Airy Equations

For the forced generalized Airy equations (3), we obtain exact particular solutions for

$$f(x) = Y_l^{(n)}(x-a), \qquad n = 0, 1, 2, ..., \ -\infty < a < \infty,. \tag{5}$$

where we assume the $Y_l(x)$, $l = 0, 1$, are known solutions of

$$\mathscr{L} Y_l(x) - x Y_l(x) = l, \qquad l = 0, 1. \tag{6}$$

Also, we indicate how to treat the cases with

$$f(x) = x^m Y_l^{(n)}(x-a), \qquad m = 1, 2, 3, ..., \tag{7}$$

and explicitly give the solution for $m = 1$. In the proofs, we make repeated use of the equation obtained by differentiating (6) r times, namely,

$$\mathscr{L} Y_l^{(r)}(x) - x Y_l^{(r)}(x) = r Y_l^{(r-1)}(x), \qquad r = 1, 2, 3, ... \tag{8}$$

We can easily prove

THEOREM 1. *A particular solution of*

$$\mathscr{L} y - xy = c Y_l^{(n)}(x), \qquad n = 0, 1, 2, ..., \tag{9}$$

where $Y_l(x)$, $l = 0, 1$, *is a solution of* (6), *is given by*

$$y(x) = \frac{c}{n+1} Y_l^{(n+1)}(x), \qquad n = 0, 1, 2, ... \tag{10}$$

Proof. Compare (9) with (8). ∎

If the right side of (9) is multiplied by x, then we have

THEOREM 2. *A particular solution of*

$$\mathscr{L}y - xy = cxY_l^{(n)}(x), \qquad n = 0, 1, 2, \ldots, \tag{11}$$

where $Y_l(x)$, $l = 0, 1$, *is a solution of* (6), *is given by*

$$y(x) = c\left[\sum_{j=0}^{k} \frac{\alpha_j}{n+j+1} Y_l^{(n+j+1)}(x) - Y_l^{(n)}(x)\right], \qquad n = 0, 1, 2, \ldots. \tag{12}$$

Proof. If $n = 0$, then using (6), (11) becomes

$$\mathscr{L}y - xy = c\mathscr{L}Y_l(x) - cl,$$

and using (4), (6), and Theorem 1, we obtain the solution (12) with $n = 0$.
If $n = 1, 2, 3, \ldots$, then using (8), (11) becomes

$$\mathscr{L}y - xy = c[\mathscr{L}Y_l^{(n)}(x) - nY_l^{(n-1)}(x)],$$

and from Theorem 1, we obtain (12). ∎

The method of proof in Theorem 2 can be used to treat $x^m Y_l^{(n)}(x)$, $m, n = 0, 1, 2, \ldots$, on the right side of (11). Using (6) and (8) repeatedly to reduce the powers of x eventually leads to a problem covered by Theorem 1.

When the forcing term in (9) has a shifted argument, i.e., $a \neq 0$ in (5), then we have

THEOREM 3. *A particular solution of*

$$\mathscr{L}y - xy = cY_l^{(n)}(x-a), \qquad n = 0, 1, 2, \ldots, a \neq 0, \tag{13}$$

where $Y_l(x)$, $l = 0, 1$, *is a solution of* (6), *is given by*

$$y(x) = \frac{n!\, c}{a}\left[\frac{l}{a^n} Y_l(x) - \sum_{j=0}^{n} \frac{Y_l^{(n-j)}(x-a)}{(n-j)!\, a^j}\right], \qquad n = 0, 1, 2, \ldots. \tag{14}$$

Proof. Let $x = z + a$, $u(z) \equiv y(z+a)$, then (13) can be rewritten in the recursion form

$$u(z) = -\frac{c}{a} Y_l^{(n)}(z) + \frac{1}{a}[\mathscr{L}u(z) - zu(z)].$$

Let $u(z) = u_0(z) + u_1(z) + \cdots + u_s(z) + \bar{u}_s(z)$. For $s = 0$, let $u_0(z) = -(c/a)\, Y_l^{(n)}(z)$, then iterating once, we obtain

$$\bar{u}_0(z) = -\frac{nc}{a^2} Y_l^{(n-1)}(z) + \frac{1}{a}[\mathscr{L}\bar{u}_0(z) - z\bar{u}_0(z)],$$

where we have used (8). We note that the order of the derivative of Y_l has been reduced by 1. Continuing these iterations, we finally obtain

$$u_n(z)+\bar{u}_n(z)=-\frac{n!\,c}{a^{n+1}}Y_l(z)-\frac{n!\,c}{a^{n+2}}[\mathscr{L}Y_l(z)-zY_l(z)] + \frac{1}{a}[\mathscr{L}\bar{u}_n(z)-z\bar{u}_n(z)];$$

so that identifying $u_n(z)=-(n!\,c/a^{n+1})\,Y_l(z)$ and using (6) we have

$$\mathscr{L}\bar{u}_n(z)-z\bar{u}_n(z)-a\bar{u}_n(z)=\frac{n!\,cl}{a^{n+1}}.$$

Noting that the right side is a constant, we easily see that

$$\bar{u}_n(z)=\frac{n!\,cl}{a^{n+1}}Y_l(z+a),$$

where we keep l in the solution since for $l=0$, $Y_0(z+a)$ is a solution of the homogeneous equation. Combining these results, we obtain (14). ▮

The cases with $x^m Y_l^{(n)}(x-a)$, $m=1,2,3,\ldots$, $a\neq 0$, on the right side of (13) can again be treated by the method of proof in Theorem 2. Letting $x=z+a$, $u(z)\equiv y(z+a)$, the transformed equation becomes

$$\begin{aligned}\mathscr{L}u(z)-zu(z)-au(z)&=c(z+a)^m\,Y_l^{(n)}(z)\\&=c\left[\sum_{i=0}^{m}\binom{m}{i}a^i z^{m-i}\right]Y_l^{(n)}(z),\end{aligned}\tag{15}$$

where $\binom{m}{i}$ are the binomial coefficients. Because of linear superposition we need only consider

$$\mathscr{L}u(z)-zu(z)-au(z)=cz^m Y_l^{(n)}(z).\tag{16}$$

Again using (6) and (8) (with x replaced by z) repeatedly to reduce the powers of z on the right, we arrive at problems covered by Theorem 3.

3. Special Forced Generalized Airy Equations

If we restrict

$$\mathscr{L}\equiv\mathscr{D}^k=\frac{d^k}{dx^k},\qquad k=1,2,3,\ldots,\tag{17}$$

the single kth-order derivative, then we can obtain further explicit results for forcing terms containing powers of x. In this section we assume $Y_l(x)$, $l=0, 1$, satisfy

$$\mathscr{D}^k Y_l(x) - xY_l(x) = l, \qquad l=0, 1. \tag{18}$$

Taking r derivatives of (18) yields

$$\mathscr{D}^k Y_l^{(r)}(x) - xY_l^{(r)}(x) = rY_l^{(r-1)}, \qquad r=1, 2, 3,.... \tag{19}$$

We then obtain the general result

THEOREM 4. *A particular solution of*

$$\mathscr{D}^k y - xy = cx^m Y_l^{(n)}(x), \qquad m, n=0, 1, 2,..., \tag{20}$$

where $Y_l(x)$, $l=0, 1$, *is a solution of* (18), *is given by*

$$y(x) = \frac{c}{km+n+1}\left[x^m Y_l^{(n+1)}(x) - kmx^{m-1}Y_l^{(n)}(x)\right] + h(x), \tag{21}$$

where

$$\begin{aligned}\mathscr{D}^k h(x) - xh(x) = {} & \frac{(k+1)\,km(m-1)\,c}{2(km+n+1)} x^{m-2} Y_l^{(k+n-1)}(x) + \cdots \\ & + \frac{(k+1)!\,(q-1)\,m!\,c}{(k-q+1)!\,q!\,(m-q)!\,(km+n+1)} x^{m-q} Y_l^{(k+n-q+1)}(x) + \cdots \\ & + \begin{cases} \dfrac{m!\,c}{km+n+1}\left[k\dbinom{k}{m-1} - \dbinom{k}{m}\right] Y_l^{(k+n-m+1)}(x) & \text{if } m \leqslant k \\[2ex] \dfrac{(k^2-1)\,m!\,c}{(m-k)!\,(km+n+1)} x^{m-k} Y_l^{(n+1)}(x) & \\ \quad - \dfrac{km!\,c}{(m-k-1)!\,(km+n+1)} x^{m-k-1} Y_l^{(n)}(x) & \text{if } m \geqslant k+1 \end{cases}\end{aligned} \tag{22}$$

Proof. Let $y(x) = c_1 x^m Y_l^{(n+1)}(x) + c_2\, mx^{m-1} Y_l^{(n)}(x) + h(x)$. Substituting into (20) and using (19), we obtain

$$
\begin{aligned}
\mathscr{D}^k y - xy = c_1 \Bigg[& (n+1)\, x^m Y_l^{(n)}(x) + mkx^{m-1} Y_l^{(k+n)}(x) \\
& + m(m-1)\binom{k}{2} x^{m-2} Y_l^{(k+n-1)}(x) + \cdots \\
& + \left\{ \begin{array}{ll} m!\dbinom{k}{m} Y_l^{(k+n-m+1)}(x) & \text{if} \quad m \leqslant k \\ \dfrac{m!}{(m-k)!} x^{m-k} Y_l^{(n+1)}(x) & \text{if} \quad m \geqslant k \end{array} \right\} \Bigg] \\
& + c_2 m \Bigg[-x^m Y_l^{(n)}(x) + x^{m-1} Y_l^{(k+n)}(x) \\
& \quad + (m-1)\, kx^{m-2} Y_l^{(k+n-1)}(x) + \cdots \\
& + \left\{ \begin{array}{ll} (m-1)!\dbinom{k}{m-1} Y_l^{(k+n-m+1)}(x) & \text{if} \quad m \leqslant k+1 \\ \dfrac{(m+1)!}{(m-k)!} kx^{m-k} Y_l^{(n+1)}(x) & \\ \quad + \dfrac{(m-1)!}{(m-k-1)!} x^{m-k-1} Y_l^{(n)}(x) & \text{if} \quad m \geqslant k+1 \end{array} \right\} \Bigg] \\
& + \mathscr{D}^k h(x) - xh(x) = cx^m Y_l^{(n)}(x).
\end{aligned}
$$

Collecting the coefficients of $x^m Y_l^{(n)}(x)$ and $x^{m-1} Y_l^{(k+n)}(x)$ and equating them to zero, we obtain $c_2 = (-kc/(km+n+1)) = -kc_1$. (For $m=0$, (21) reduces to (10) with $h(x) \equiv 0$.) Combining the remaining terms, we obtain (21). On the right side of (22) the power of x has been reduced by at least 1, depending on the value of k. Continuing this procedure, the power of x is eventually reduced to zero and Theorem 1 applies. ∎

Including a shift in the argument of Y_l in (20) is a special case of Theorem 3. Including powers of x times $Y_l^{(n)}(x-a)$ can be treated by the method of proof in Theorem 2, as mentioned previously.

For completeness, we consider the special cases where the forcing functions are given by x^m. Let $L_k(x)$ be a solution of

$$\mathscr{D}^k L_k(x) - xL_k(x) = 1, \qquad k = 1, 2, 3, \ldots. \tag{23}$$

Then we have

LEMMA. *A particular solution of*

$$\mathscr{D}^k y - xy = cx^m, \qquad m = 0, 1, \ldots, k; \quad k = 1, 2, 3, \ldots, \tag{24}$$

is given by

$$\begin{aligned} y(x) &= cL_k(x) && \text{if} \quad m=0, \\ &= -cx^{m-1} && \text{if} \quad m=1,2,\dots,k. \end{aligned} \tag{25}$$

Proof. For $m=0$, compare (23) and (24). For $m=1,2,\dots,k$, since $\mathscr{D}^k x^{m-1}=0$, the result follows. ▌

For general m, including the above Lemma, we have

THEOREM 5. *A particular solution of*

$$\mathscr{D}^k y - xy = cx^m, \qquad m=0,1,2,\dots; \quad k=1,2,3,\dots, \tag{26}$$

is given by

$$y(x) = -c\left\{\sum_{j=0}^{[m/(k+1)]-1} \begin{bmatrix} m-1 \\ m+1-j(k+1) \end{bmatrix}_{k+1} x^{m-j(k+1)-1} + \begin{bmatrix} -\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_{k+1} L_k(x) & \text{if} \quad m=p(k+1) \\ \begin{bmatrix} m-1 \\ q+1 \end{bmatrix}_{k+1} x^{q-1} & \text{if} \quad m=p(k+1)+q \end{bmatrix}\right\}, \tag{27}$$

where $p=0,1,2,\dots$; $q=1,2,\dots,k$, $[m/(k+1)]=$ *greatest integer* $\leqslant m/(k+1)$, *and*

$$\begin{aligned} \begin{bmatrix} m-1 \\ m-\mu \end{bmatrix}_{k+1} &\equiv \prod_{i=1}^{(\mu+1)/(k+1)} \frac{(m-1-(i-1)(k+1))!}{(m-i(k+1))!} && \text{if} \quad 1 \leqslant m-\mu, \\ &\equiv 1 && \text{if} \quad \mu<1, \end{aligned} \tag{28}$$

where $(\mu+1)/(k+1)$ *is an integer.*

Proof. For $m=0,1,\dots,k$, see the Lemma. For $m=k+1$, $k+2,\dots$, let $y(x)=-cx^{m-1}+h(x)$ and substitute into (26) giving

$$\mathscr{D}^k h(x) - xh(x) = c\frac{(m-1)!}{(m-k-1)!}x^{m-k-1} = c\begin{bmatrix} m-1 \\ m-k \end{bmatrix}_{k+1} x^{m-k-1}.$$

This has the same form as (26) with the power of x reduced by $k+1$. Therefore, the procedure can be repeated and eventually terminated at one of the "primitive" problems covered by the Lemma. ▌

Appendix

Equation (1) arises from a consideration of the boundary-value problem for the differential-difference equation

$$y''(x) - xy(x) + \alpha y'(x-a) + \beta y(x-a) = 0, \qquad 0 < x < l, \quad 0 < a < l, \tag{A.1}$$

$$y(x) = \phi = \text{constant} \qquad \text{for} \quad -a \leqslant x \leqslant 0, \quad y(l) = \gamma. \tag{A.2}$$

On the interval $0 < x < a$, y satisfies the equation

$$y'' - xy = -\beta\phi, \tag{A.3}$$

which has the general solution

$$y(x) = b_1 Ai(x) + b_2 Bi(x) - \beta\phi L(x), \tag{A.4}$$

where b_1, b_2 are constants; Ai, Bi are Airy functions; and L is a Lommel function (see (2) in Section 1). Then, on the interval $a < x < \min(2a, l)$, y satisfies an equation of the form given by (1).

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